Small Oscillation (contd.)

Let us now discuss the eigen vectors for three modes of vibration.

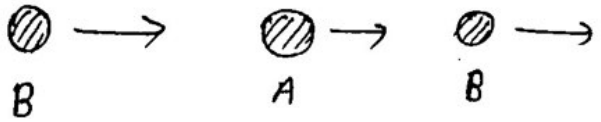
(i) For $\omega_1 = 0$

$$\begin{pmatrix} K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = 0$$

on solving, $a_{11} = a_{21} = a_{31} = \alpha$ (say)

So, the eigen vector is given by

$$a_1 = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} \qquad \text{——— (6)}$$



(Antisymmetric mode)

i.e. for $\omega_1 = 0$, all the molecules suffer same displacement in same direction.

(ii) For $\omega_2 = \sqrt{K/m}$

$$\begin{pmatrix} 0 & -K & 0 \\ -K & 2K - \frac{MK}{m} & -K \\ 0 & -K & 0 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = 0$$

on solving we have

$$a_{22} = 0$$

$$-a_{12} - a_{32} = 0$$

Therefore, $a_{22} = 0$, $a_{12} = -a_{32} = \beta$ (say)

Thus, for $\omega_2 = \sqrt{K/m}$, the eigen vector is given by

$$a_2 = \begin{pmatrix} \beta \\ 0 \\ -\beta \end{pmatrix} \qquad \text{——— (7)}$$

(symmetric mode)

(iii) For $\omega_3 = \sqrt{\frac{K}{m}\left(1 + \frac{2m}{M}\right)}$

$$\begin{pmatrix} -\frac{2mK}{M} & -K & 0 \\ -K & -\frac{KM}{m} & -K \\ 0 & -K & \frac{-2mK}{M} \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = 0$$

On solving we get—

$$\frac{2m}{M} a_{13} + a_{23} = 0$$

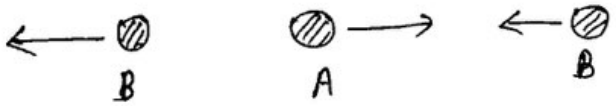
$$a_{13} + \frac{m}{M} + a_{33} = 0$$

$$a_{23} + \frac{2m}{M} a_{33} = 0$$

Therefore, $a_{13} = a_{33} = \gamma$ (say) and $a_{23} = \left(\frac{-2m}{M}\right)\gamma$

Thus, the eigen vector is given by

$$a_3 = \begin{pmatrix} \gamma \\ \frac{-2m}{M}\gamma \\ \gamma \end{pmatrix} \qquad \text{——— (8)}$$



(Antisymmetric mode)

Evaluation of α, β & γ

Matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & \frac{-2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{pmatrix}$

$$A^T = \begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & \frac{-2m}{M}\gamma & \gamma \end{pmatrix}$$

We impose the matrix condition $A^T A = I$

$$\text{or} \quad \begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & \frac{-2m}{M}\gamma & \gamma \end{pmatrix} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & -m \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & \frac{-2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} (2m+M)\alpha^2 & 0 & 0 \\ 0 & 2m\beta^2 & 0 \\ 0 & 0 & 2m\gamma^2\left(1+\frac{2m}{M}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{or} \quad (2m+M)\alpha^2 = 1$$

$$\therefore \quad \alpha = \sqrt{1/(2m+M)}$$

Similarly, $\beta = \sqrt{1/2m}$

$$\gamma = \sqrt{\frac{1}{2m\left(1+\frac{2m}{M}\right)}}$$

Eigenvectors :—

From equations (6), (7) & (8), we get—

$$a_1 = \frac{1}{\sqrt{2m+M}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$a_2 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$a_3 = \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$

These are called normal co-ordinates.

In general,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

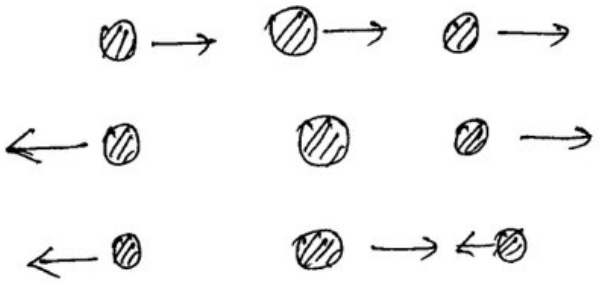
$$\therefore x_i = \sum_{k=1}^{3} a_{ik} q_k$$

Where, $i = 1, 2, 3.$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2m+M}} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} \\ \frac{1}{\sqrt{2m+M}} & 0 & \frac{-2m}{M}\Big/\sqrt{2m\left(1+\frac{2m}{M}\right)} \\ \frac{1}{\sqrt{2m+M}} & \frac{-1}{\sqrt{2m}} & \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Conclusion



Longitudinal normal modes of triatomic molecules.